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An Energy Approach to Static Analysis of a Centrally Loaded Simply Supported Thin Rectangular Isotropic Plate

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ABSTRACT: In this paper, the numerical factors for deflection of a centrally loaded all round simply supported thin rectangular isotropic plate are determined using energy approach. The results obtained using energy approaches are compared with those obtained from literature. It is shown that results obtained using energy method are agreed favorably with those of other author serving as the exact solution showing the efficacy of the energy method in the determination of numerical factors for deflection of a centrally loaded simply supported thin rectangular isotropic plate. It is also found that the value of the numerical factors for deflection of all round simply supported thin rectangular isotropic plate is a function of the assumed polynomial function.

Keywords: Numerical Factors, Rectangular Isotropic Plate, Exact Solution, Energy Approach, Polynomial Function

ORIGINAL ARTICLE

INTRODUCTION

A plate is defined as an engineering structure whose thickness is small compared with its other dimensions. Plates such as slabs are used in modern engineering structures to transmit lateral and or in-plane load to adjacent support. Plates are employed in engineering construction, building, civil engineering, hydraulic engineering, naval architecture and air-craft construction (Biot, 1972; Iyengar, 1988; Chajes, 1974). The classical method that leads to exact solution is not only rigorous and time consuming but proves in many cases quite laborious and almost impossible due to its mathematical difficulties (Gould, 1999; Fenner, 1986). The problems encountered in thin plate theory can be solved with the aid of various approximate methods such as energy method, finite element method, finite difference method and Fourier series (El Naschie, 1990; Vinson, 1974). However, common problems are encountered. For example, numerical finite difference and finite element methods lead to an algebraic equation of large matrix size which requires large computer memories, thereby making the analysis cumbersome and time wasting. The static analysis of all round simply supported plates is made possible by levy (Timoshenko and Woinowsky, 1959). Nevertheless, the double trigonometric series involved in the method are not convenient for numerical computations especially when higher derivatives of the deflection function are involved. To overcome the mathematical difficulties and other shortcomings involved in the static analysis of all round simply supported thin rectangular isotropic plate subjects to centre-point loading using both classical and numerical approaches, an energy method is employed in this study as a tool to estimate the numerical factors for deflection of all round simply supported thin rectangular isotropic plate. The algorithm involved is simple and straightforward. It is believed that research into the use of energy approach to static analysis of all round simply supported thin rectangular isotropic plate will enhance maximum utilization of plates as engineering structures.

Derivation of energy equations of equilibrium

When a system is in a state of static equilibrium, its total energy is minimum.

$$\Rightarrow \delta\pi_T = 0 \quad (1)$$

For a body that behaves elastically, the potential energy can be written as:

$$\pi_T = \int_R W dR - \int_{st} T_i U_i d_s \quad (2)$$

Where:
W= strain energy
R = volume of elastic body
T_i= ith component of surface traction
U_i= ith component of deformation
s_t =portion of the surface over which traction is prescribed.

From equations (1) and (2), we have:

$$\delta\pi_T = \delta\left[\int_R W dR - \int_{s_t} T_i U_i ds\right] = 0 \quad (3)$$

According to Timoshenko (1959), the strain energy in an elastic plate is given by:

$$U = \frac{D}{2}\int_o^{lx}\int_o^{ly}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dxdy \quad (4)$$

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The potential energy of applied load is given by:

$$V = -pw(x,y) \quad (5)$$

and the total potential energy of the system is given by:

$$\pi_T = U + V \quad (6)$$

$$\pi_T = \int_o^{lx}\int_o^{ly}\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

Substituting equations (4) and (5) into equation (6) gives:

$$dxdy - \int_o^{lx}\int_o^{ly} pw(x,y)dxdy \quad (7)$$

Let $\quad w(x,y) = A.\varphi(x).\psi(y) \quad (8)$

be the solution of equation (7)
Where:

- $\varphi(x)$ = Represent x-coordinate
- $\psi(y)$ = Represent y-coordinate
- A = coordinate defining the shape of the deflection surface

Let

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (9)$$

$$\psi(y) = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \quad (10)$$

By differentiating, we have the following derivatives:
For x – direction:

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (11)$$

$$\varphi^{/}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \quad (12)$$

$$\varphi^{//}(x) = 2a_2 + 6a_3 x + 12a_4 x^2 \quad (13)$$

$$\varphi^{///}(x) = 6a_3 + 24a_4 x \quad (14)$$

Likewise, for y – direction:

$$\psi(y) = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \quad (15)$$

$$\psi^{/}(y) = b_1 + 2b_2 y + 3b_3 y^2 + 4b_4 y^3 \quad (16)$$

$$\psi^{//}(y) = 2b_2 + 6b_3 y + 12b_4 y^2 \quad (17)$$

$$\psi^{///}(x) = 6b_3 + 24b_4 y \quad (18)$$

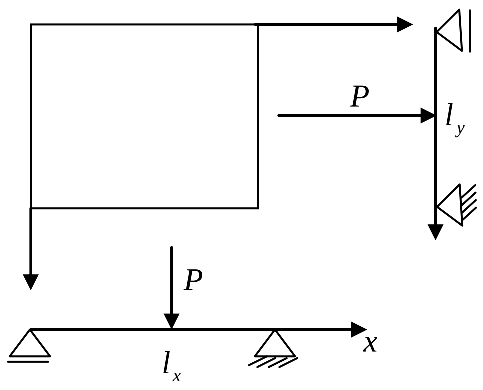

Figure 1. All round simply supported plate subjected to centre point loading.

The boundary conditions are:

$$\left.\begin{array}{lll}\varphi_{(0)} = 0 & \varphi_{(l_x)} = 0 & \varphi_{\left(\frac{l_x}{2}\right)} = 1 \\ \varphi^{/}{}_{(0)} = 0 & \varphi^{/}{}_{(l_x)} = 0 & \end{array}\right\} \quad (19)$$

At x = 0:

$$\left.\begin{array}{ll}\varphi_{(0)} = 0 & = a_0 \\ \varphi^{/}{}_{(0)} = 0 & = a_1\end{array}\right\} \quad (20)$$

At x = l_x:

$$\left.\begin{array}{l}\varphi = 0 = a_0 + a_1 l_x + a_2 l_x^2 + a_3 l_x^3 + a_4 l_x^4 \\ \varphi^{/} = 0 = a_1 + 2a_2 l_x + 3a_3 l_x^2 + 4a_4 l_x^3\end{array}\right\} \quad (21)$$

At x = $\frac{l_x}{2}$:

$$\varphi_{\left(\frac{l_x}{2}\right)} = 1 = a_0 + a_1\frac{l_x}{2} + \frac{a_2 l_x^2}{4} + \frac{a_3 l_x^3}{8} + \frac{a_4 l_x^4}{16} \quad (22)$$

Substituting equation (20) into (21) gives:

$$0 = 6a_3 l_x + 12a_4 l_x^2$$
$$a_3 = 2a_4 l_x \quad (23)$$

Substituting equation (20) and (23) into (21a) gives:

$$0 = 0 + a_1 l_x + 0 - 2a_4 l_x^4 + a_4 l_x^4$$
$$a_1 = a_4 l_x^3 \quad (24)$$

Substituting equation (20), (23), (24) into (22) gives:

$$1 = 0 + \frac{a_4 l_x^4}{2} + 0 - \frac{2a_4 l_x^4}{8} + \frac{a_4 l_x^4}{16} = \frac{5a_4 l_x^4}{16} \quad (25)$$
$$a_4 = \frac{16}{5l_x^4}$$

Substituting equation (20) into (23) gives:

$$a_3 = \frac{2\times 16}{5l_x^4}\times l_x = -\frac{32}{5l_x^3} \quad (26)$$

Substituting equation (25) and (24) gives:

$$a_1 = \frac{16}{5l_x^4}\times l_x^3 = \frac{16}{5l_x} \quad (27)$$

Therefore,

$$a_0 = 0, a_1 = \frac{16}{5l_x},\ a_2 = 0,\ \ a_3 = \frac{32}{5l_x^3},\ a_4 = \frac{16}{5l_x^4} \quad (28)$$

Using equation (28), equations (9) and (10) become:

$$\varphi(x) = \frac{16x}{5l_x} - \frac{32x^3}{5l_x^3} + \frac{16x^4}{l_x^4} \quad = \frac{16}{5}\left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \quad (29)$$

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Similarly,

$$\psi(y)=\frac{16y}{5l_y}-\frac{32y^3}{5l_y^3}+\frac{16y^4}{5l_y^4}\quad=\frac{16}{5}\left(\frac{y}{l_y}-\frac{2y^3}{l_y^3}+\frac{y^4}{l_y^4}\right) \qquad (30)$$

Substituting equation (29) and (30) into (8) we have:

$$w(x,y)=A.\frac{16}{5}\left(\frac{x}{l_x}-\frac{2x^3}{l_x^3}+\frac{x^4}{l_x^4}\right).\frac{16}{5}\left(\frac{y}{l_y^2}-\frac{2y^3}{l_y^3}+\frac{y^4}{l_x^4}\right)$$

$$A\frac{256}{25}\left(\frac{x}{l_x}-\frac{2x^3}{l_x^3}+\frac{x^4}{l_x^4}\right).\left(\frac{y}{l_y}-\frac{2y^3}{l_y^3}+\frac{y^4}{l_x^4}\right) \qquad (31)$$

Therefore,

$$\frac{\partial w}{\partial x^2}=A\frac{256}{25}\left(\frac{l}{l_x}-\frac{6x^2}{l_x^3}+\frac{4x^3}{l_x^4}\right).\left(\frac{y}{l_y}-\frac{2y^3}{l_y^3}+\frac{y^4}{l_x^4}\right)$$

$$\frac{\partial^2 w}{\partial x^2}=A\frac{256}{25}\left(-\frac{12x}{l_x^3}+\frac{12x^2}{l_x^4}\right).\left(\frac{y}{l_y}-\frac{2y^3}{l_y^3}+\frac{y^4}{l_x^4}\right) \qquad (32)$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2}=A.256\left(\frac{-12y}{l_y^3}+\frac{12y^2}{l_y^4}\right)\left(\frac{x}{l_x}-\frac{2x^3}{l_x^3}+\frac{x^4}{l_x^4}\right) \qquad (33)$$

Expansion of equation (32) gives:

$$\frac{\partial^2 w}{\partial x^2}=A\frac{256}{25}\left(\frac{-12xy}{l_x^3 l_y}+\frac{24xy^3}{l_x^3 l_y^3}-\frac{12xy^4}{l_x^3 l_y^4}+\frac{12x^2y}{l_x^3 l_y}-\frac{24x^2y^3}{l_x^4 l_y^3}+\frac{12x^2y^4}{l_x^4 l_y^4}\right)$$

$$\frac{\partial^2 w}{\partial x^2}=A\frac{3072xy}{25l_x l_y}\left(-\frac{1}{l_y^2}+\frac{2y^2}{l_x^2 l_y^2}-\frac{y^3}{l_x^2 l_y^3}+\frac{x}{l_x^3}-\frac{2xy^2}{l_x^3 l_y^2}+\frac{xy^3}{l_x^3 l_y^3}\right) \qquad (34)$$

Therefore,

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2=A^2\left(\frac{3072xy}{25l_x l_y}\right)^2\left(\frac{1}{l_x^4}-\frac{4y^2}{l_x^4 l_y^2}+\frac{2y^3}{l_x^4 l_y^3}-\frac{2x}{l_x^5}+\frac{8xy^2}{l_x^5 l_y^2}-\frac{4xy^3}{l_x^5 l_y^3}+\frac{4y^4}{l_x^4 l_y^4}-\frac{4y^5}{l_x^4 l_y^5}\right.$$
$$-\frac{8xy^4}{l_x^5 l_y^4}+\frac{8xy^5}{l_x^5 l_y^5}+\frac{y^6}{l_x^4 l_y^6}-\frac{2xy^6}{l_x^5 l_y^6}+\frac{x^2}{l_x^6}+\frac{4x^2y^2}{l_x^6 l_y^2}+\frac{2x^2y^3}{l_x^6 l_y^3}+\frac{4x^2y^4}{l_x^6 l_y^4}$$
$$\left.-\frac{4x^2y^5}{l_x^6 l_y^5}+\frac{x^2y^6}{l_x^6 l_y^6}\right) \qquad (35)$$

Similarly,

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^2=A^2\left(\frac{3072xy}{25l_x l_y}\right)^2\left(\frac{1}{l_x^4}-\frac{4x^2}{l_x^2 l_y^4}+\frac{2x^3}{l_x^3 l_y^4}-\frac{2y}{l_y^5}+\frac{8x^2y}{l_x^2 l_y^5}-\frac{4x^3y}{l_x^3 l_y^5}+\frac{4x^4}{l_x^4 l_y^4}-\frac{4x^5}{l_x^5 l_y^4}\right.$$
$$-\frac{8x^4y}{l_x^4 l_y^5}+\frac{8x^5y}{l_x^5 l_y^5}+\frac{x^6}{l_x^4 l_y^6}-\frac{2x^6y}{l_x^5 l_y^6}+\frac{x^2}{l_x^6}+\frac{4x^2y^2}{l_x^2 l_y^6}+\frac{2x^3y^2}{l_x^3 l_y^6}+\frac{4x^4y^2}{l_x^4 l_y^4}$$
$$\left.-\frac{4x^5y^2}{l_x^5 l_y^6}+\frac{x^6y^2}{l_x^6 l_y^6}\right) \qquad (36)$$

Applying the strain energy equation for thin rectangular isotropic plate, given as:

$$U=\int_0^{l_x}\int_0^{l_y}\frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dxdy$$

According to Gould [4], the third term in the strain energy equation is negligible for plates in which the plane form is polygonal and the edges remain straight. The strain energy equation above now reduces to:

$$U=\int_0^{l_x}\int_0^{l_y}\frac{D}{2}\left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right)dxdy \qquad (37)$$

Substituting the derivatives of equations (35) and (38) into (37) and integrating, gives:

$$U=\frac{DA^2}{2}\left(\frac{3072}{25l_x l_y}\right)^2\left[\frac{31}{18900}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right)\right]=\frac{DA^2 12.383184}{l_x^2 l_y^2}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right) \qquad (38)$$

Substituting equation (31) into (5) the potential energy V due to the external load becomes:

$$V=-\frac{P.A.256\zeta\eta}{25l_x l_y}\beta_2 \qquad (39)$$

Where

$$\beta_2=1-\frac{2\eta^2}{l_y^2}+\frac{\eta^3}{l_y^3}-\frac{2\zeta^2}{l_x^2}+\frac{4\zeta\eta^2}{l_x^2 l_y^2}-\frac{2\zeta^2\eta^2}{l_x^3 l_y^3}+\frac{\zeta^3}{l_x^3}-\frac{2\zeta^3\eta^2}{l_x^3 l_x^2}+\frac{\zeta^3\eta^3}{l_x^3 l_y^3}$$

The total potential energy of plate is:

$$\pi_T=U-V \qquad (40)$$

Substituting equations (38) and (39) into (40) making equation it a minimum yields:

$$\frac{\partial\Pi_T}{\partial A}=0$$

$$\Rightarrow\ \frac{2.D.A12.383184}{l_x^2 l_y^2}\left(\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right)-\frac{P256\zeta\eta}{25l_x l_y}\beta_2=0$$

Therefore,

$$A=\frac{P256\zeta\eta l_x l_y}{D24.766366\left[\frac{l_y^3}{l_x}+\frac{l_x^3}{l_y}\right]\times 25}\beta_2 \qquad (41)$$

Substitution of equation (41) into (8) yields:

$$w(x,y)=\frac{0.413463969P\zeta\eta l_x l_y\ \ \beta_2}{D\left[\frac{l_y^3}{l_x}+\frac{31l_x^3}{6l_y}\right]}\varphi(x)\psi(y) \qquad (42)$$

Equation (42) gives the numerical factors for deflection of an all round simply supported thin rectangular isotropic plate given different values of span ratio:

RESULTS AND DISCUSSION

A numerical example is given in this study to demonstrate the applicability of energy method in the determination of numerical factors for deflection of all round simply supported thin rectangular isotropic plate subjected to centre point.

Taking the maximum deflection of the plate to be at the middle when the point load is applied at the middle.

Then we have $\xi=x={}^{l_x}\!/_2,\ \eta=y={}^{l_y}\!/_2$.

Table 1. Comparison of results of numerical factors α for deflection of a centrally loaded simply supported rectangular plate for various span ratio l_y/l_x .

l_y/l_x	**Energy method**	**Levy's method (Timoshenlo and Woinowsky-Kriger, 1959)**
	ASS	ASS
1.0	$\frac{0.090445243P}{D}$	$\frac{0.1856P}{D}$
1.1	$\frac{0.097709199P}{D}$	$\frac{0.2024P}{D}$
1.2	$\frac{0.10169793P}{D}$	$\frac{0.21648P}{D}$
1.3	$\frac{0.103061746P}{D}$	$\frac{0.22696P}{D}$
1.4	$\frac{0.10452055P}{D}$	$\frac{0.23744P}{D}$
1.5	$\frac{0.10470192P}{D}$	$\frac{0.24432P}{D}$

The comparison of results of numerical factors for deflection of all round simply supported thin rectangular isotropic plate subjected to transverse point load using energy method are compared with those of the exact solutions (Timoshenlo and Woinowsky-Kriger, 1959) as shown in Table 1. From Table 1, it can be seen that the numerical factors for deflection increase as the span ratio increases which is in agreement with those levy (Timoshenlo and Woinowsky-Kriger, 1959) serving as the exact solution. The results obtained using energy methods are quite close to those of the exact solution. The disparity between the results obtained using energy method and those of levy serving as the exact solution may be attributed to the assumed polynomial function which is not too close to the actual deflection function as levy's single trigonometric function is. Increasing the number of terms in the assumed polynomial function may yield better results.

CONCLUSION

From the study the following conclusions are drawn.

- The deflection co-efficient increase as the span ratio increases in both cases.
- The energy method produced results that are almost identical with those of levy's showing the efficacy of the energy method in the determination of numerical factors for deflection of all round simply supported plates.
- Even though energy method has proved to be an excellent approximate method for determination of numerical coefficients for deflection of all round simply supported plate, this study has proved that unsatisfactory results can be obtained if an unsatisfactory deflection function is assumed.

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